

3.5 p.1

MA121

May 28/18

Implicit Differentiation and Related Rates

We have seen functions expressed as the dependent variable in terms of an independent variable.

ie $y = f(x)$ or $p = f(q)$ These forms are called explicit relationships. However, it is also very common to have the dependent variable and the independent variable intermingling in an equation. These forms are called implicit relationships.

ie $x^2 + y^2 = 25$ or $p^2 + 2pq + 3q^2 = 16$
we can still find the slope functions $\frac{dy}{dx}$ in implicit relations. This process is called implicit differentiation.

Reversing the chain rule given a composite function, $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x))g'(x)$
We apply the chain rule in implicit differentiation. We can differentiate an equation with respect to any variable.

Example given $x^2 + y^2 = 25$

i) we can differentiate this equation with respect to x

* derivative of a constant is always 0

$$x^2 + y^2 = 25$$

$$\Rightarrow 2x \frac{dx}{dx} + 2y \frac{dy}{dx} = 0$$

Explanation: $\frac{dy}{dx} \leftarrow$ this means we differentiate an expression in y
 $\frac{dx}{dx} \leftarrow$ with respect to x .

$\frac{dx}{dx} \leftarrow$ differentiate an expression in x

$\frac{dx}{dx} \leftarrow$ with respect to x

3.5 p2

$$\begin{aligned} \Rightarrow 2x + 2y \frac{dy}{dx} &= 0 \leftarrow \text{Since } \frac{dx}{dx} = 1 \\ &= 2y \frac{dy}{dx} = -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= \frac{dy}{dx} = -\frac{x}{y} \end{aligned}$$

We could also differentiate this equation with respect to y from $x^2 + y^2 = 25$.

$$\Rightarrow 2x \frac{dx}{dy} + 2y \frac{dy}{dy} = 0$$

$\frac{dx}{dy} \leftarrow$ differentiate an expression in x
 $\frac{dy}{dy} \leftarrow$ with respect to y
 $\frac{dy}{dy} \leftarrow$ differentiate an expression in y
 $\frac{dy}{dy} \leftarrow$ with respect to y.

Continuing,

$$\begin{aligned} 2x \frac{dx}{dy} + 2y \frac{dy}{dy} &= 0 \\ \Rightarrow 2x \frac{dx}{dy} &= -2y \\ \frac{dx}{dy} &= \frac{-2y}{2x} \end{aligned}$$

$$\frac{dx}{dy} = -\frac{y}{x}$$

Observations

i) Consider $y = x^2 + 2x - 1$

$$\begin{aligned} \frac{dy}{dx} &= 2x \frac{dx}{dx} + 2 \frac{dx}{dx} - 0 \\ \frac{dy}{dx} &= 2x + 2 \end{aligned}$$

ii) Since $x^2 + y^2 = 25$ is an equation with one independent variable, we can see that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

iii) $\frac{dx}{dy}$ and $\frac{dy}{dx}$ behave like quotients.

If we have an equation in p and q, we have only one independent variable, $\frac{dp}{dq} = \frac{1}{\frac{dq}{dp}}$ or $\frac{dq}{dp} = \frac{1}{\frac{dp}{dq}}$

Ch 3 p3

MATH

May 28/18

We could also differentiate $x^2 + y^2 = 25$ with respect to any other variable. From $x^2 + y^2 = 25$,

$$\Rightarrow 2x \frac{dx}{dy} + 2y \frac{dy}{dx} = 0 \quad \frac{dx}{dt} \leftarrow \text{differentiate an expression}$$

$$\Rightarrow x \frac{dx}{dy} + y \frac{dy}{dx} = 0 \quad \frac{dx}{dt} \leftarrow \text{with respect to time } t \text{ in } x$$

Example

2p90

$$xy = 7(x+y)^4 \leftarrow \text{we want to find } y' = \frac{dy}{dx}$$

ie) we want to differentiate the equation with respect to x

$$\begin{aligned} \text{So, we have } yx' + xy' &= 28(x+y)^3(x+y)' \\ \Rightarrow y\left(+\frac{dx}{dx}\right) + x\left(1\frac{dy}{dx}\right) &= 28(x+y)^3\left(1\frac{dx}{dx} + 1\frac{dy}{dx}\right) \\ \Rightarrow y + x\frac{dy}{dx} &= 28(x+y)^3\left(1 + \frac{dy}{dx}\right) \\ \Rightarrow y + x\frac{dy}{dx} &= 28(x+y)^3 + 28(x+y)^3\frac{dy}{dx} \\ \Rightarrow x\frac{dy}{dx} - 28(x+y)^3\frac{dy}{dx} &= 28(x+y)^3 - y \\ \Rightarrow (x - 28(x+y)^3)\frac{dy}{dx} &= 28(x+y)^3 - y \\ = \frac{dy}{dx} &= \frac{28(x+y)^3 - y}{x - 28(x+y)^3} \end{aligned}$$

Example

8p90

$$\frac{p+1}{q-1} = p+q \leftarrow \text{we want to find } p' \frac{dp}{dq}$$

ie) we want to differentiate the equation with respect to q . from $\frac{p+1}{q-1} = p+q$

$$\begin{aligned} \Rightarrow \frac{(q-1)(p+1)' - (p+1)(q-1)'}{(q-1)^2} &= p' + q' \\ \Rightarrow \frac{(q-1)\left(1\frac{dp}{dq}\right) - (p+1)\left(1\frac{dq}{dq}\right)}{(q-1)^2} &= 1\frac{dp}{dq} + 1\frac{dq}{dq} \\ \Rightarrow \frac{(q-1)\frac{dp}{dq} - (p+1)}{(q-1)^2} &= \frac{dp}{dq} + 1 \\ \Rightarrow (q-1)\frac{dp}{dq} - p - 1 &= (q-1)^2\left(\frac{dp}{dq} + 1\right) \end{aligned}$$

Wrt = with respect to

p4 Ch 3

$$\Rightarrow \frac{(q-1) \frac{dp}{dq}}{(q-1)^2} - \frac{p+1}{(q-1)^2} = \frac{dp}{dq} + 1$$

$$\Rightarrow \frac{1}{q-1} \frac{dp}{dq} - \frac{p+1}{(q-1)^2} = \frac{dp}{dq} + 1$$

$$\Rightarrow \frac{1}{q-1} \frac{dp}{dq} = 1 + \frac{p+1}{(q-1)^2}$$

$$\Rightarrow \left(\frac{1}{q-1} - 1 \right) \frac{dp}{dq} = 1 + \frac{p+1}{(q-1)^2}$$

$$\Rightarrow \left(\frac{1-(q-1)}{q-1} \right) \frac{dp}{dq} = \frac{(q-1)^2 + p+1}{(q-1)^2}$$

$$\Rightarrow \frac{2-q}{q-1} \frac{dp}{dq} = \frac{(q-1)^2 + p+1}{(q-1)^2}$$

$$\Rightarrow \frac{dp}{dq} = \left(\frac{q-1}{2-q} \right) \left(\frac{(q-1)^2 + p+1}{(q-1)^2} \right)$$

$$\Rightarrow \frac{dp}{dq} = \frac{(q-1)^2 + p+1}{(2-q)(q-1)}$$

EXAMPLE 11 p 90 Find the rate p is changing with respect to time, t .
(e) $\frac{dp}{dt}$ where $\sqrt{p+q} = 10p$, and $q = 99$, $p = \$1$ and $\frac{dq}{dt} = -1$

differentiating implicitly,

$$(p+q)^{1/2} = 10p \text{ wrt } t$$

$$\Rightarrow \frac{1}{2}(p+q)^{-1/2} (p+q)' = 10p'$$

$$\Rightarrow \frac{1}{2\sqrt{p+q}} \left(\frac{dp}{dt} + \frac{dq}{dt} \right) = 10 \frac{dp}{dt} \quad \leftarrow q = 99, p = \$1, \frac{dq}{dt} = -1$$

$$\Rightarrow \frac{1}{2\sqrt{1+99}} \left(\frac{dp}{dt} + (-1) \right) = 10 \frac{dp}{dt}$$

$$\Rightarrow \frac{1}{2\sqrt{100}} \left(\frac{dp}{dt} - 1 \right) = 10 \frac{dp}{dt}$$

$$\Rightarrow \frac{1}{20} \left(\frac{dp}{dt} - 1 \right) = 10 \frac{dp}{dt}$$

$$\Rightarrow \frac{dp}{dt} - 1 = (20) \left(10 \frac{dp}{dt} \right) = 200 \frac{dp}{dt}$$

$$\Rightarrow 200 \frac{dp}{dt} - \frac{dp}{dt} = -1$$

$$\Rightarrow 199 \frac{dp}{dt} = -1$$

$$\Rightarrow \frac{dp}{dt} = \frac{-1}{199}$$

$$\approx \$0.01$$

May 28/8

ma-121

Ch 3 ps

13p 90

Find the equation of the tangent line to $y^2 + y = 2x$ at the point $(x, y) = (55, 10)$.
First, find the slope function $\frac{dy}{dx}$, differentiate with respect to x .

From $y^2 + y = 2x$

$$\Rightarrow (2y + 1) \frac{dy}{dx} = 2 \frac{dx}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{2y+1}$$

At $(x, y) = (55, 10)$, the slope of the tangent line is $m = \frac{2}{2(10)+1} = \frac{2}{21}$

The equivalent of this tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 10 = \frac{2}{21}(x - 55)$$

$$y - 10 = \frac{2}{21}x - \frac{110}{21}$$

$$y = \frac{2}{21}x - \frac{110}{21} + 10$$

$$= y = \frac{2}{21}x + \frac{100}{21}$$

2018/11

15:44

2018/11/15

10/11